# Solving RED with Weighted Proximal Methods

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# Outline

# Inverse Problems – Optimization Problem

Image Denoising - "Simplest" Inverse:

$$\underbrace{\boldsymbol{y}}_{\text{Measured}} = \underbrace{\boldsymbol{x}}_{\text{Clean}} + \underbrace{\boldsymbol{n}}_{\text{Noise (AWGN)}}$$

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Maximum a Posteriori Probability (MAP) – Prior

Optimization Problem

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \underbrace{\ell(\boldsymbol{x}, \boldsymbol{y})}_{\text{Data Fidelity}} + \lambda \underbrace{\rho(\boldsymbol{x})}_{\text{Prior}}$$

$\ell(\boldsymbol{x}, \boldsymbol{y})$ : linear or nonlinear $\rightarrow \frac{1}{2\sigma^2} \|\boldsymbol{x} - \boldsymbol{y}\|_2^2$ or $\frac{1}{2\sigma^2} \|\boldsymbol{Hx} - \boldsymbol{y}\|_2^2$

$\rho(\cdot)$ : TV, sparsity, low-rank, CNN etc.

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*Finding effective* $\rho(\cdot)$?

# What is RED? – REgularized by Denoising


image denoising

About 123'000 results (**0.05** sec)

A non-local algorithm for **image denoising**
A Buades, B Coll, JM Morel - 2005 IEEE Computer Society …, 2005 - ieeexplore.ieee.org
We propose a new measure, the method noise, to evaluate and compare the performance of digital **image denoising** methods. We first compute and analyze this method noise for a wide class of **denoising** algorithms, namely the local smoothing filters. Second, we propose a new …
Cited by 4996 Related articles All 24 versions

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$$\rho(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathcal{T}}(\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x}))$$

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How to minimize $\phi(\boldsymbol{x})$? It is weird $\rightarrow \boldsymbol{f}(\boldsymbol{x})$.

# The Properties of RED

## Assumptions:

- Differentiability: $\boldsymbol{f}(\boldsymbol{x}) : [0,1]^n \rightarrow [0,1]^n$
- Local Homogeneity: $\boldsymbol{f}(c\boldsymbol{x}) = c\boldsymbol{f}(\boldsymbol{x}),$ if $|c-1| \leq \varepsilon \ll 1$
- Passivity: $\|\boldsymbol{f}(\boldsymbol{x})\| \leq \|\boldsymbol{x}\|$

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$$\begin{aligned} \nabla_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{x} &= \lim_{\varepsilon -> 0} \frac{\boldsymbol{f}(\boldsymbol{x}+\varepsilon\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})}{\varepsilon} \\ &= \lim_{\varepsilon -> 0} \frac{(1+\varepsilon)\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})}{\varepsilon} \\ &= \boldsymbol{f}(\boldsymbol{x}) \end{aligned}$$

$$\frac{\partial\left(\frac{1}{2}\boldsymbol{x}^{\mathcal{T}}\left(\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})\right)\right)}{\partial \boldsymbol{x}} = \boldsymbol{x} - \frac{1}{2}\boldsymbol{f}(\boldsymbol{x}) - \frac{1}{2}\nabla_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{x} = \boxed{\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})}$$

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Conclusions:

- $\rho(\boldsymbol{x})$ is convex. If $\ell(\boldsymbol{x}, \boldsymbol{y})$ is convex, $\phi(\boldsymbol{x})$ is convex.
- Evaluate one time gradient or $\phi(\boldsymbol{x})$, call one time $\boldsymbol{f}(\boldsymbol{x})$.

# How Many Algorithms Satisfy these Assumptions?

[REM17]: We have many, some of them are state-of-the-art.

NLM, Bilateral, kernal regression, TNRD etc.

Others $\varepsilon$-modified: Median, K-svd, BM3D, EPLL, CNN etc.

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- gradient based methods: gradient descent/Nesterov Acceleration, conjugate gradient, BFGS, LBFGS etc. – line search? [NW06]

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- Accelerated Proximal Gradient (APG) [RS19]
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In practice: APG $\geq$ VE $>$ FP $>$ ADMM $>$ gradient based

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Weighted Proximal Methods can do better. :-)

# Development of the Proximal Gradient Methods

Composite problem:

$$\min_{\boldsymbol{x}} \phi(\boldsymbol{x}) \triangleq g(\boldsymbol{x}) + h(\boldsymbol{x})$$

$g(\boldsymbol{x})$: convex, differentiability
$h(\boldsymbol{x})$: convex, can be nonsmooth
The solution is nonempty.

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$\boldsymbol{x}_k$: solution at $k$th iteration
Linearizing $g(\boldsymbol{x})$ at $\boldsymbol{x}_k$ [1]:

$$g(\boldsymbol{x}) + h(\boldsymbol{x}) \leq \hat{\phi}(\boldsymbol{x}, \boldsymbol{x}_k) \triangleq g(\boldsymbol{x}_k) + < \nabla_{\boldsymbol{x}} g(\boldsymbol{x}_k), \boldsymbol{x} - \boldsymbol{x}_k > + \frac{L}{2} \|\boldsymbol{x} - \boldsymbol{x}_k\|_2^2 + h(\boldsymbol{x})$$

$$\nabla_{\boldsymbol{x}}^2 g(\boldsymbol{x}) \preceq L$$

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Minimize $\hat{\phi}(\boldsymbol{x}, \boldsymbol{x}_k)$ instead of minimizing $\phi(\boldsymbol{x})$ at $(k+1)$th iteration:

$$Prox_{\frac{1}{L}h}(\boldsymbol{v}_k) = \arg\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{x} - \boldsymbol{v}_k\|_2^2 + \frac{1}{L} h(\boldsymbol{x}) : \text{Closed-Form}$$

$$\boldsymbol{v}_k = \boldsymbol{x}_k - \frac{1}{L} \nabla_{\boldsymbol{x}} g(\boldsymbol{x}_k)$$

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# Acceleration – Nesterov/FISTA

Set $\boldsymbol{y}_1 = \boldsymbol{x}_0$ and $t_1 = 1$ and repeat the following at step $k \geq 1$

- $\boldsymbol{x}_k = Prox_{\frac{1}{L}h}(\boldsymbol{y}_k - \frac{1}{L}\nabla_{\boldsymbol{x}} g(\boldsymbol{y}_k))$
- $\boxed{t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}}$
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Convergence Speed:

Proximal: $\mathcal{O}(\frac{1}{k})$

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Can we do better?

No closed-form —— $Prox_{\frac{1}{L}h}(\cdot)$ —— RED

# Weighted Proximal Methods

Linearizing $g(\boldsymbol{x})$ with higher accuracy:

$$g(\boldsymbol{x}) + h(\boldsymbol{x}) \leq \hat{\phi}(\boldsymbol{x}, \boldsymbol{x}_k)$$

$$\hat{\phi}(\boldsymbol{x}, \boldsymbol{x}_k) \triangleq g(\boldsymbol{x}_k) + < \nabla_{\boldsymbol{x}} g(\boldsymbol{x}_k), \boldsymbol{x} - \boldsymbol{x}_k > + \frac{1}{2a_k} \boxed{(\boldsymbol{x} - \boldsymbol{x}_k)^{\mathcal{T}} \boldsymbol{B}_k (\boldsymbol{x} - \boldsymbol{x}_k)} + h(\boldsymbol{x})$$

$a_k$ stepsize or use 1 and $\boldsymbol{B}_k \succ 0$. Define

$$Prox_{a_k h}^{WPM}(\boldsymbol{v}_k) = \arg\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{x} - \boldsymbol{v}_k\|_{\boldsymbol{B}_k}^2 + a_k h(\boldsymbol{x}) : \text{No Closed-Form}$$

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In RED:

Remind the denoising $\boldsymbol{f}(\boldsymbol{x})$ in RED: high complexity

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Remind the denoising $\boldsymbol{f}(\boldsymbol{x})$ in RED: high complexity

To reduce the calling of $\boldsymbol{f}(\boldsymbol{x})$, we set

$$\boxed{g(\boldsymbol{x}) = \lambda \rho(\boldsymbol{x})}$$

and

$$\boxed{h(\boldsymbol{x}) = \ell(\boldsymbol{x}, \boldsymbol{y})}$$

# The Choice of $\boldsymbol{B}_k$ – $\underbrace{\ell(\boldsymbol{x}, \boldsymbol{y})}_{h(\boldsymbol{x})} + \underbrace{\frac{\lambda}{2}\boldsymbol{x}^{\mathcal{T}}\left(\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})\right)}_{g(\boldsymbol{x})}$

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Proximal method:

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- $\boldsymbol{B}_k = \lambda\boldsymbol{I}$ and $a_k = 1$: recover proximal method
- $\boldsymbol{B}_k$: the Hessian of $g(\boldsymbol{x}) \rightarrow$ Quasi-Newton [NW06]
- ...

# Estimate $\boldsymbol{B}_k$ — SR1

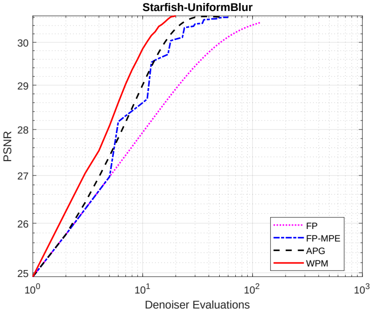
**Algorithm 1** SR1

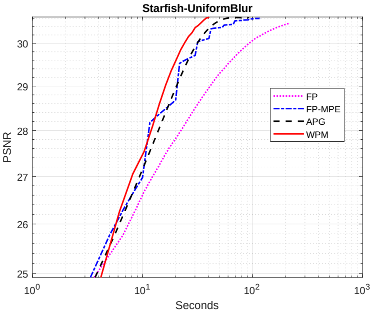
**Initialization:** $k = 1$, $\gamma = 1.25$, $\delta = 10^{-8}$, $\boldsymbol{x}_k$, $\boldsymbol{x}_{k-1}$, $\nabla g(\boldsymbol{x}_k)$, $\nabla g(\boldsymbol{x}_{k-1})$.

```
if k = 1 then
    B_k ← αI
else
    Set [s_k] ← x_k − x_{k−1} and [m_k] ← ∇g(x_k) − ∇g(x_{k−1})
    Calculate τ ← γ ‖m_k‖_2^2 / ⟨s_k, m_k⟩
    if τ < 0 then
        [B_k ← αI]
    else
        [H_0] ← τI
        if |⟨m_k − H_0 s_k, s_k⟩| ≤ δ‖s_k‖_2 ‖m_k − H_0 s_k‖_2 then
            u_k ← 0
        else
            u_k ← (m_k − H_0 s_k) / sqrt(⟨m_k − H_0 s_k, s_k⟩)
        end if
        [B_k ← H_0 + u_k u_k^T]
    end if
end if
Return: B_k
```

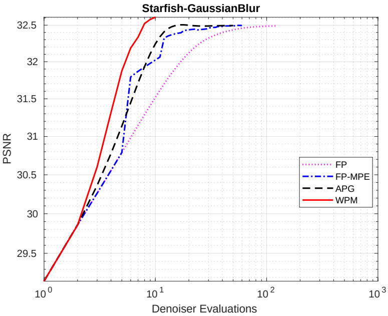
Where: line 4: $\boldsymbol{s}_k \leftarrow \boldsymbol{x}_k - \boldsymbol{x}_{k-1}$, $\boldsymbol{m}_k \leftarrow \nabla g(\boldsymbol{x}_k) - \nabla g(\boldsymbol{x}_{k-1})$; line 5: $\tau \leftarrow \gamma \frac{\|\boldsymbol{m}_k\|_2^2}{\langle \boldsymbol{s}_k, \boldsymbol{m}_k \rangle}$; line 13: $\boldsymbol{u}_k \leftarrow \frac{\boldsymbol{m}_k - \boldsymbol{H}_0 \boldsymbol{s}_k}{\sqrt{\langle \boldsymbol{m}_k - \boldsymbol{H}_0 \boldsymbol{s}_k, \boldsymbol{s}_k \rangle}}$; line 15: $\boldsymbol{B}_k \leftarrow \boldsymbol{H}_0 + \boldsymbol{u}_k \boldsymbol{u}_k^{\mathcal{T}}$

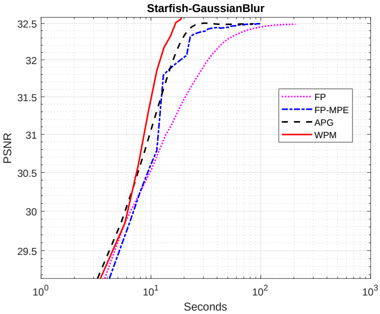
# Image Deblurring - Uniform



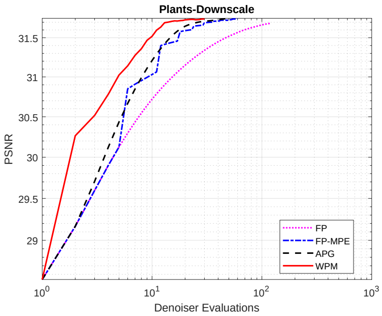



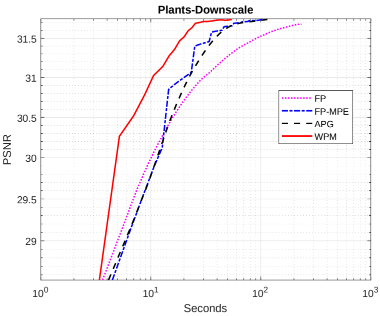
# Image Deblurring - Gaussian



# Image Super-Resolution (SR)



# More Results [HYZ19]

FP – 200 denoiser evaluations

Denoiser evaluations, other methods – compariable PSNR

1st and 2nd row: deblurring with uniform and Gaussian blurs.

3rd row: SR

| | FP-MPE | APG | WPM |
|---|---|---|---|
| Butterfly | 54 | 34 | 25 |
| | 54 | 26 | 17 |
| | 80 | 51 | 26 |
| Boats | 24 | 20 | 21 |
| | 60 | 34 | 22 |
| | 36 | 20 | 12 |
| House | 24 | 18 | 19 |
| | 62 | 26 | 25 |
| | 18 | 15 | 10 |
| Parrot | 39 | 30 | 20 |
| | 52 | 40 | 36 |
| | 49 | 31 | 28 |
| Lena | 48 | 34 | 29 |
| | 47 | 16 | 15 |
| | 37 | 26 | 18 |
| Barbara | 14 | 12 | 11 |
| | 48 | 23 | 16 |
| | 17 | 15 | 11 |
| Peppers | 42 | 29 | 22 |
| | 41 | 40 | 34 |
| | 38 | 30 | 28 |
| Leaves | 50 | 41 | 34 |
| | 36 | 18 | 14 |
| | 60 | 41 | 12 |

# Conclusion

*WPMs are good if no closed-form solution exists for the proximal operator.*

Thanks & Questions?

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